



Noncommutative integrability and recursion operators[☆]

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Abstract

Geometric structures underlying commutative and noncommutative integrable dynamics are analyzed. They lead to a new characterization of noncommutative integrability in terms of spectral properties and of Nijenhuis torsion of an invariant $(1,1)$ tensor field. The construction of compatible symplectic structures is also discussed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the last few decades there has been a renewed interest in completely integrable Hamiltonian systems, whose concept goes back to the last century [25,26] and which, loosely speaking, are dynamical systems admitting a Hamiltonian description and possessing sufficiently many constants of motion, so that they can be integrated by quadratures. Some qualitative features of these systems remain true in some special classes of infinite-dimensional Hamiltonian systems expressed by nonlinear evolution equations as, for instance, Korteweg–de Vries and sine-Gordon [36].

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A relevant progress in the study of these systems with an infinite-dimensional phase manifold \mathcal{M} was the introduction of the *Lax representation* [23] which played an important role in formulating the *inverse scattering method*, universally recognized as one of the most remarkable result of theoretical physics in last decades, and of the *AKNS scheme* [1]. This method allows the integration of nonlinear dynamics, both with finitely or infinitely many degrees of freedom, for which a Lax representation can be given [17], this being both of physical and mathematical relevance [34].

Most of the evolution equations admitting a Lax representation are generally Hamiltonian dynamics on infinite-dimensional *weakly symplectic* manifolds so that the natural arena, for the analysis of their integrability, is represented by the phase space with its natural symplectic structure. In terms of this structure the scattering data associated to the Lax operator has a natural interpretation as *action–angle* type variables [38].

A further progress, in the analysis of the integrability, was the important remark that many of the previous systems are Hamiltonian dynamics with respect to two *compatible* symplectic structures [18,27,28,36], this leading to a geometrical interpretation of the so-called *recursion operator* [23].

This fact suggested that the integrability of nonlinear field theories could be naturally explained in terms of mixed tensor fields, whose relation [23] with Lax operators is still [10,11] unclear.

As a matter of fact, a description of integrability [10,13,14,16,21,24,28], which does not depend in a crucial way on dimensionality, and so works both for systems with finitely many degrees of freedom and for field theory can be given in terms of invariant mixed tensor field, having bidimensional eigenspaces and vanishing Nijenhuis torsion [33].

The analysis of the integrability realized with the help of such tensor field leads to the formulation of an integrability criterion [10,13,14,24] which, for finite-dimensional systems, is *essentially* equivalent to the classic Liouville theorem.

To be more specific, the mentioned *essential equivalence* means that the equivalence holds for *nonresonant* Hamiltonian systems, i.e. for completely integrable dynamics whose Hamiltonian expressed in action–angle coordinates has a nonvanishing Hessian.

One reason for a completely integrable Hamiltonian system to be resonant may be that the number of first integrals, defined on the entire phase space, is larger than one half of the phase space dimension (of course, in this case not all the integrals are in involution and one will have to deal with noncommuting sets of first integrals). This happens for the Kepler dynamics which, however, is bi-Hamiltonian and has a recursion operator with the right properties [29].

More in general, the analysis of symmetries [20] shows that generally one is faced with a non-Abelian algebra corresponding, for Hamiltonian systems, to a non-Abelian algebra of first integrals.

The integrability of such systems with finitely many degrees of freedom has been analyzed in several papers [31]. There exist field dynamics, related to vector and matrix nonlinear Schroedinger equation [19,22], possessing a noncommutative set of first integrals so that it would be useful to have a noncommutative integrability criterion formulated in terms of a recursion operator. In this paper such a criterion is presented. More specifically, in

Section 2 integrability criteria for the commutative case are recalled. Section 3 is devoted to noncommutative integrability. After a review of known results a new noncommutative integrability criterion is presented. The Kepler dynamics is discussed as an example.

2. Commutative integrability criteria

The best known criterion of integrability goes back to the celebrated Liouville theorem [25,26] and can be reported [2,3,5,30,35] as follows.

Theorem 1. *If on a $2n$ -dimensional symplectic manifold \mathcal{M} are defined a Hamiltonian dynamics and n functionally independent of first integrals f_1, \dots, f_n in involution*

$$\{f_i, f_j\} = 0 \quad \forall i, j = 1, \dots, n,$$

whose associated Hamiltonian fields X_i are complete, then the level manifolds

$$\mathcal{M}_{f(\pi)} = \{p \in \mathcal{M} : f_i(p) = \pi_i, i = 1, \dots, n\}$$

are invariant with respect to the dynamics and each of their connected components is diffeomorphic either to $T^m \times \mathfrak{R}^{n-m}$ or, if compact, to a torus T^n . Moreover, for every point $p \in \mathcal{M}$ near which m is constant, there exists a neighborhood \mathcal{U} invariant under the composed flow of the vector fields X_i , and canonical coordinates $(P_1, \dots, P_n, Q^1, \dots, Q^n)$, where Q^1, \dots, Q^m are angles, such that the equation of motion takes the form

$$\dot{P}_i = 0, \quad \dot{Q}^i = v^i(P), \quad 1 \leq i \leq n.$$

A more general setting for the commutative integrability is the following [24,37].

Let \mathcal{M} be a smooth $2n$ -dimensional manifold. Let us suppose we can find n vector fields $X_1, \dots, X_n \in \chi(\mathcal{M})$ and n functions $f_1, \dots, f_n \in \mathcal{F}(\mathcal{M})$ with the following properties:

$$[X_i, X_j] = 0, \tag{1}$$

$$L_{X_i} f^j = 0, \quad i, j \in \{1, \dots, n\}. \tag{2}$$

It can be shown that, if on an open dense submanifold of \mathcal{M} ,

$$X_1 \wedge \dots \wedge X_n \neq 0, \tag{3}$$

$$df^1 \wedge \dots \wedge df^n \neq 0, \tag{4}$$

any dynamical system Δ on \mathcal{M} which is of the form

$$\Delta = \sum_{i=1}^n v^i X_i, \quad v^i = v^i(f^1, \dots, f^n), \tag{5}$$

is completely integrable on the submanifold in which Eqs. (3) and (4) are satisfied.

If the fields X_i are complete, by using the n -functions f^1, \dots, f^n , a family of symplectic structures can be defined with respect to which the dynamics is Hamiltonian.

In the Liouville theorem [25,26] only the commuting first integrals and the symplectic structure are given. Of course, the commuting vector fields are constructed from them. An alternative integrability theorem, suggested by the analysis of integrable models in field theory, can be formulated [10,13,14] using invariant tensor fields and it reads as follows.

Theorem 2 (DMSV). *Let Δ be a dynamical vector field on a differential manifold \mathcal{M} which admits a (1,1) mixed tensor field T which*

- *is invariant:*

$$L_{\Delta}T = 0,$$

- *has a vanishing Nijenhuis torsion:*

$$\mathcal{N}_T = 0,$$

- *is diagonalizable with doubly degenerate eigenvalues λ_j whose differentials $d\lambda_j$ are independent at each point.*

Then, the vector field Δ is separable, completely integrable and Hamiltonian.

Remark 3. *We observe that the Hamiltonian character of the dynamics Δ is not assumed a priori but it follows from the properties of the tensor field T , so that all dynamics, satisfying the given hypotheses, result to be Liouville integrable. Integrability of dissipative dynamics can be put in the same setting by assuming [12] different spectral hypotheses for the tensor field T .*

The last formulation has the advantage of being more appropriate to deal with dynamics with infinitely many degrees of freedom (completely integrable field theories). We also observe that the Lax representation, the powerful integration tool for such systems, may not be useful in more than one space dimension since the inverse problem in *quantum mechanics* has been solved only for one-dimensional systems.

2.1. From the Liouville integrability to invariant mixed tensor fields

Let us now study the problem of constructing invariant mixed tensor fields, with the appropriate properties (also called a *recursion tensor field*), for a given Liouville's integrable Hamiltonian dynamics Δ . If H is the Hamiltonian function and $\{\cdot, \cdot\}$ denotes the Poisson bracket, we have

$$\Delta f = \{H, f\}.$$

Let us introduce in some neighborhood of a Liouville's torus T^n action-angle coordinates $(J_1, \dots, J_n, \varphi^1, \dots, \varphi^n)$, in which we have

$$\omega = \sum_h dJ_h \wedge d\varphi^h, \quad \Delta = \frac{\partial H}{\partial J_h} \frac{\partial}{\partial \varphi^h}.$$

Let us distinguish two cases.

1. The Hamiltonian H is a separable one:

$$H = \sum_k H_k(J_k).$$

In this case a class of recursion tensor fields can be easily defined

$$T = \sum_h \lambda_h(J_h) \left(dJ_h \otimes \frac{\partial}{\partial J_h} + d\varphi^h \otimes \frac{\partial}{\partial \varphi^h} \right)$$

with the λ 's arbitrary and functionally independent. Indeed, the tensor field T is invariant and has vanishing Nijenhuis torsion and doubly degenerate eigenvalues.

2. The Hamiltonian has a nonvanishing Hessian:

$$\det \left(\frac{\partial^2 H}{\partial J_h \partial J_k} \right) \neq 0.$$

In this case, in the chosen neighborhood, setting

$$v^h(J) = \frac{\partial H}{\partial J_h},$$

new coordinates (v/φ) can be introduced, so that the dynamics can be described with respect to the new symplectic structure:

$$\omega_1 = \sum_h dv^h \wedge d\varphi^h = \sum_{hk} \frac{\partial^2 H}{\partial J_h \partial J_k} dJ_k \wedge d\varphi^h$$

by a separable Hamiltonian function

$$H_1 = \frac{1}{2} \sum_h (v^h)^2.$$

As before, a class of recursion tensor fields is then given by

$$T = \sum_h \lambda_h(v^h) \left(dv^h \otimes \frac{\partial}{\partial v^h} + d\varphi^h \otimes \frac{\partial}{\partial \varphi^h} \right).$$

By means of this construction it is possible to find the second symplectic structure for a completely integrable Hamiltonian system. It is still an open problem if this is true also in the remaining cases. In this direction one may find useful hints in [6,7,15]. The next section is concerned with noncommutative integrable dynamics and, in particular, with their characterization in terms of an invariant, mixed tensor field.

3. Noncommutative integrability criteria

As it has been observed in Section 1, if the number of independent first integrals is larger than half the dimension of the symplectic manifold, they cannot be in involution anymore and one will have to deal with noncommuting sets of first integrals. For a finite number of

degrees of freedom a noncommutative generalization of Liouville theorem is the following [4,31].

Theorem 4 (MF1). *A Hamiltonian vector field on a symplectic manifold (\mathcal{M}, ω) having a noncommutative Lie algebra \mathcal{A} of first integrals satisfying the condition*

$$\dim \mathcal{A} + \text{rank } \mathcal{A} = \dim \mathcal{M},$$

where the rank of \mathcal{A} is the maximum of the rank of the matrix $m_{ab} = \{f_a, f_b\}$,¹ is completely integrable, i.e. the joint level surfaces of the first integrals are invariant, and in a neighborhood of each invariant surface one can define canonical coordinates $(\lambda/\chi/p/q)$, the χ 's being the coordinates on the invariant surfaces, such that Hamilton's equations take the form

$$\dot{\lambda}_i = 0, \quad \dot{\chi}^i = v_i, \quad \dot{P}_\alpha = 0, \quad \dot{q}^\alpha = 0, \quad 1 \leq i \leq r, \quad r+1 \leq \alpha \leq n$$

with $r = \text{rank } \mathcal{A}$. If these invariant surfaces are compact and connected, one can prove, as in the commutative case, that they are tori, and the χ 's can be chosen to be angle variables. The canonical coordinates are called in this case "generalized action–angle variables".

The Liouville theorem can be recovered [5] as stated by the following.

Theorem 5 (MF2). *If \mathcal{M} is compact, then under the hypotheses of Theorem 4, one can find $n = \frac{1}{2} \dim \mathcal{M}$ first integrals which are in involution.*

Even in this case, however, the noncommutative theorem, showing the full symmetry of the system, remains of interest. A full account of the relevant geometrical structures underlying the noncommutative integrability can be found in [8,9,15,32]. Here we just give a short review of them.

A symplectic form on \mathcal{M} at a point p defines a skewsymmetric bilinear nondegenerate form on $T_p\mathcal{M}$. If W is a r -dimensional subspace of $T_p\mathcal{M}$, the symplectic orthogonal subspace $W^\perp \equiv \{X \in T_p\mathcal{M} : \omega(X, Y) = 0 \forall Y \in W\}$ has dimension $2n - r$ and in general $W \cap W^\perp \neq 0$.

Two cases are of particular interest: $r \leq n$ and $r \geq n$. If $r \leq n$ and $W \subseteq W^\perp$, W is said to be *isotropic*, and if $r \geq n$ and $W \supseteq W^\perp$, W is called *coisotropic*. If W is isotropic and coisotropic ($r = n$), then it is called *Lagrangian*. A submanifold is called *isotropic*, *coisotropic* or *Lagrangian* if its tangent spaces are isotropic, coisotropic or Lagrangian, respectively.

In the commutative case the level surfaces of the first integrals f_i define an invariant Lagrangian foliation \mathcal{F}_1 of \mathcal{M} . The Hamiltonian vector fields X_i associated to the functions f_i are then a basis of commuting tangent vector fields for the leaves and can be used to define local coordinates χ^i on the leaves. These fields also commute with the Hamiltonian vector field Δ which, consequently, can be expressed as $\Delta = v^i(f)X_i$. In a neighborhood

¹ For semisimple Lie algebras, this definition coincides with the usual one.

of a point $p \in \mathcal{M}$, the set (χ/f) defines canonical coordinates and Hamilton's equations of motion take the simple following form:

$$\dot{\chi}^i = v^i, \quad \dot{f}_i = 0.$$

In the noncommutative case the first integrals f_a , $1 \leq a \leq 2n - r$, still define an invariant foliation, but the leaves now have dimension $r \leq n$ and the Hamiltonian vector fields X_a , associated with the first integrals f_a , are not all tangent to the leaves. However, the condition $\dim \mathcal{A} + \text{rank } \mathcal{A} = \dim \mathcal{M}$ ensures for each leaf l , the existence of a subalgebra \mathcal{A}_l which commutes with \mathcal{A} on l . The Hamiltonian vector fields \bar{X}_i , associated to a basis of \mathcal{A}_l , will give themselves a basis of tangent vector fields for l and will have the property $\omega(\bar{X}_i, X_a)|_l = 0$, so that each leaf will be isotropic.² To obtain a set of canonical coordinate, in a neighborhood of a point of l and eventually of the whole of l , one needs to exploit further properties of this isotropic foliation. At each point p of l consider the subspace $T_p l \subseteq T_p \mathcal{M}$ and the resulting distribution of symplectically orthogonal subspaces $(T_p l)^\perp$. Since $\omega(\bar{X}_i, X_a)|_l = 0$, this distribution is generated for all leaves, by the vector fields X_a , and, furthermore, since X_a satisfy the hypotheses of the Frobenius theorem, we obtain a second coisotropic foliation \mathcal{F}_2 whose leaves are themselves foliated by those of the first foliation \mathcal{F}_1 . The regularity of this foliation follows from the independence of the functions f_a . One can now prove the existence of canonical coordinates $(\lambda_i, \chi^i, p_\alpha, q^\alpha)$, such that the symplectic structure and the dynamical vector field take the following form:

$$\omega = d\lambda_i \wedge d\chi^i + dp_\alpha \wedge dq^\alpha, \quad \Delta = v^i(\lambda)X_i,$$

so that the equations of motion become

$$\dot{\lambda}_i = 0, \quad \dot{\chi}^i = v^i, \quad \dot{p}_\alpha = 0, \quad \dot{q}^\alpha = 0.$$

The functions λ_i describe locally \mathcal{F}_2 , and their associated Hamiltonian vector fields X_i define coordinates χ^i on \mathcal{F}_1 . The fields X_i are independent and, since $\omega(X_i, X_\alpha) = d\lambda_i(X_\alpha) = 0$, they are tangent to the leaves of \mathcal{F}_1 , and thus commute among themselves and with Δ . To understand better this canonical coordinate, one can actually observe that the momentum map $J : \mathcal{M} \rightarrow \mathcal{A}^*$ is defined by $J : x \rightarrow \xi_x \in \mathcal{A}^*$, where $\xi_x(f) \equiv f(x)$, $f \in \mathcal{A}$, defines a fibration of a neighborhood \mathcal{U} of a leaf of \mathcal{F}_2 with fiber $l_x = J^{-1}(\xi_x)$, namely a leaf of \mathcal{F}_1 . The neighborhood \mathcal{U} can then be represented as $l_x \times S \times \mathcal{O}$, where \mathcal{O} is a region in the coadjoint orbit through ξ_x of the Lie group corresponding to \mathcal{A} and S is a linear manifold transverse to \mathcal{O} . The symplectic structure ω restricted to \mathcal{O} coincides with the Lie–Kirillov–Kostant–Souriau symplectic form; (p_α, q^α) are canonical coordinates on \mathcal{O} and λ_i coordinates on S . It has been actually proved [15] that all what is needed for the existence of such local canonical coordinates is the double foliation, namely that \mathcal{M} has an isotropic foliation such that the distribution of subspaces, symplectically orthogonal to the tangent spaces to its leaves, is integrable.

² In particular, $\omega(X_i, X_j)|_l = 0$.

3.1. Noncommutative integrability and invariant tensor field

Let us give now the following new characterization of noncommutative integrability.

Theorem 6. *Let Δ be a dynamical vector field on a $2n$ -differential manifold \mathcal{M} which admits a $(1,1)$ mixed tensor field T , which*

- *is invariant:*

$$L_{\Delta}T = 0,$$

- *is diagonalizable with only simple and doubly degenerate eigenvalues whose differentials are independent at each point $p \in \mathcal{M}$,*
- *has the property*

$$\mathcal{N}_T(\alpha, X, Y) = 0 \quad \forall X : X(p) \in S(p), Y \in \mathcal{D}(\mathcal{M}),$$

and for all 1-forms α , $S(p)$ denoting the sum of eigenspaces associated to the doubly degenerate eigenvalues of $T(p)$.

Then, the vector field Δ is separable, completely integrable and Hamiltonian.

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the doubly degenerate eigenvalues and $\mu_{2r+1}, \dots, \mu_{2n}$ be the simple ones. Under the hypotheses, the tensor field T can be written in the form

$$T = \sum_{i=1}^r \lambda_i (e_i \otimes \vartheta^i + e_{i+r} \otimes \vartheta^{i+r}) + \sum_{\alpha=2r+1}^{2n} \mu_{\alpha} e_{\alpha} \otimes \vartheta^{\alpha}, \tag{6}$$

where the e 's form a basis of eigenvectors of T and the ϑ 's are the elements of the dual basis. Thus,

$$\begin{aligned} Te_i &= \lambda_i e_i, & Te_{i+r} &= \lambda_i e_{i+r}, & Te_{\alpha} &= \mu_{\alpha} e_{\alpha}, & i \leq r, \alpha \geq 2r + 1, \\ T\vartheta^i &= \lambda_i \vartheta^i, & T\vartheta^{i+r} &= \lambda_i \vartheta^{i+r}, & T\vartheta^{\alpha} &= \mu_{\alpha} \vartheta^{\alpha}, & i \leq r, \alpha \geq 2r + 1. \end{aligned} \tag{7}$$

The Nijenhuis torsion [33] of T , defined by

$$\mathcal{N}_T(\alpha, X, Y,) = \langle \alpha, \mathcal{H}_T(X, Y) \rangle \tag{8}$$

with

$$\mathcal{H}_T(X, Y) = [TX, TY] + T^2[X, Y] - T[TX, Y] - T[X, TY], \tag{9}$$

once evaluated on the basis vector fields $\{e_1, \dots, e_{2n}\}$, gives

$$\begin{aligned} \mathcal{H}_T(e_i, e_j) &= (T - \lambda_i)(T - \lambda_j)[e_i, e_j] + (\lambda_i - \lambda_j)[(L_{e_i} \lambda_j)e_j + (L_{e_j} \lambda_i)e_i] \\ \mathcal{H}_T(e_i, e_{\alpha}) &= (T - \lambda_i)(T - \mu_{\alpha})[e_i, e_j] + (\lambda_i - \mu_{\alpha})[(L_{e_i} \mu_{\alpha})e_{\alpha} + (L_{e_{\alpha}} \lambda_i)e_i], \end{aligned}$$

where $i, j \leq 2r$ and $\alpha \geq 2r + 1$, so that the conditions on the torsion imply the following relations:

$$\begin{aligned} (T - \lambda_i)(T - \lambda_j)[e_i, e_j] &= 0, & (\lambda_i - \lambda_j)e_i(\lambda_j) &= 0, \\ (T - \lambda_i)(T - \mu_{\alpha})[e_i, e_{\alpha}] &= 0, & e_i(\mu_{\alpha}) &= e_{\alpha}(\lambda_i) = 0. \end{aligned} \tag{10}$$

It follows that for any three vector fields e_i, e_j, e_α ,

$$[e_i, e_j] = ae_i + be_j + ce_{i+r} + de_{j+r}, \quad [e_i, e_\alpha] = fe_i + ge_{i+r} + he_\alpha.$$

Thus, the two vector fields e_i and e_{i+r} , belonging to the same eigenvalue λ_i , satisfy the relation

$$[e_i, e_{i+r}] = c_i e_i + c_{i+r} e_{i+r}. \quad (11)$$

Therefore, $\forall i \in \{1, \dots, r\}$, the vector fields e_i, e_{i+r} are a local basis of a two-dimensional involutive distribution and, by Frobenius' theorem, define a two-dimensional submanifold of \mathcal{M} . In other words, they can be chosen so that, on each bi-dimensional manifold, coordinates ξ^i, η^i can be found such that

$$e_i = \frac{\partial}{\partial \xi^i}, \quad e_{i+r} = \frac{\partial}{\partial \eta^i}. \quad (12)$$

In conclusion, relations (10), which directly follows from the Nijenhuis condition, ensure the “partial” holonomicity of the basis, in which the tensor field T is diagonal.

On the other hand, since

$$d\lambda_i = \vartheta^j e_j(\lambda_i) + \vartheta^\alpha e_\alpha(\lambda_i) = \vartheta^j e_j(\lambda_i), \quad (13)$$

we have, by using Eq. (10),

$$T d\lambda_i = T \vartheta^j e_j(\lambda_i) = \vartheta^j \lambda_j e_j(\lambda_i) = \vartheta^j \lambda_i e_j(\lambda_i) = \lambda_i d\lambda_i. \quad (14)$$

Moreover,

$$d\mu^\rho \equiv d\mu_\rho = \sum_{k=1}^{2r} \vartheta^k e_k(\mu_\rho) + \sum_{\alpha=1}^{2n} \vartheta^\alpha e_\alpha(\mu_\rho) = \sum_{\alpha=1}^{2n} \vartheta^\alpha e_\alpha(\mu_\rho).$$

By means of the above relations, it is now possible to choose a holonomic basis in such a way that T has the following expression:

$$T = \sum_{j=1}^r \lambda_j (e_j \otimes \vartheta^j + e_{r+j} \otimes d\lambda^j) + C_\rho^\sigma e_\sigma \otimes d\mu^\rho \quad (15)$$

with

$$C_\rho^\sigma = \sum_{\alpha=2r+1}^{2n} \mu_\alpha e_\alpha(\mu^\sigma) [e_\alpha(\mu^\rho)]^{-1}, \quad \vartheta^i = 0.$$

In addition, in a neighborhood of each bi-dimensional submanifold we can choose coordinates $(\lambda/\chi/\mu)$ such that the tensor T can also be written in the form³

$$T = \sum_{j=1}^r \lambda_j \left(\frac{\partial}{\partial \lambda_i} \otimes d\lambda_i + \frac{\partial}{\partial \chi^i} \otimes d\chi^i \right) + C_\rho^\sigma \frac{\partial}{\partial \mu_\rho} \otimes d\mu_\sigma.$$

³ The symbols used for the coordinates have been chosen just to correspond to the geometric structures previously described.

In the chosen basis, the vector field Δ can be written as

$$\Delta = \Lambda_i \frac{\partial}{\partial \lambda_i} + \Phi^i \frac{\partial}{\partial \chi^i} + E^\alpha e_\alpha, \quad (16)$$

so that the condition $L_\Delta T = 0$ implies that $\Lambda_i = E^\alpha = 0$. It follows that

$$\Delta = \Phi^i(\lambda_i, \chi^i) \frac{\partial}{\partial \chi^i}. \quad (17)$$

Symplectic structures can be found with respect to which the above vector field is Hamiltonian. Indeed, the closed 2-form

$$\omega = \sum_{k=1}^r G_k(\lambda_k, \chi^k) d\lambda_k \wedge d\chi^k + \sum_{\alpha, \beta=2r+1}^{2n} f_{\alpha\beta}(\mu_\alpha, \mu_\beta) d\mu_\alpha \wedge d\mu_\beta$$

will be invariant if

$$\frac{\partial}{\partial \chi^i} (G_i \Phi^i) = 0.$$

The nondegeneracy condition for ω is obviously expressed by

$$\det \|f_{\alpha\beta}\| \prod_{k=1}^r G_k \neq 0.$$

This is equivalent to require that if $\Phi^i(\lambda_i, \chi^i)$ vanishes at some point then it also vanishes on the whole integral curve of $\partial/\partial \chi^i$ through that point. If the vector field Δ has no singular points,⁴ a particularly simple class of symplectic structures with respect to which it is Hamiltonian is given by

$$\omega = \sum_{k=1}^r \frac{g_k(\lambda_k)}{\Phi^k(\lambda_k, \chi^k)} d\lambda_k \wedge d\chi_k + \sum_{\alpha, \beta=2r+1}^{2n} f_{\alpha\beta}(\mu_\alpha, \mu_\beta) d\mu_\alpha \wedge d\mu_\beta,$$

where g_k and $f_{\alpha\beta}$ are arbitrary functions such that

$$\det \|f_{\alpha\beta}\| \prod_{k=1}^r \frac{g_k}{\Phi^k} \neq 0.$$

⁴ If Φ_k is identically zero for some index k , we can define

$$\omega = \sum_i g_i(\lambda_i) d\lambda_i \wedge d\chi_i + \sum_j \frac{g_j(\lambda_j)}{\Phi^j(\lambda_j, \chi_j)} d\lambda_j \wedge d\chi_j + \sum_{\alpha, \beta=2r+1}^{2n} f_{\alpha\beta}(\mu_\alpha, \mu_\beta) d\mu_\alpha \wedge d\mu_\beta,$$

where the sum on the index i runs over those eigenspaces for which $\Phi^j = 0$. When Δ has zeroes but does not vanish identically, we have to exclude this closed subset from our consideration. These sets will be invariant under the flow so that our analysis can be carried over in the same fashion as we have done on the complement.

If the submanifold $\mu=\text{constant}$ is compact and connected, we can introduce, as usual, action–angle coordinates (J, φ) so that the vector field Δ and the symplectic structure ω , in the coordinates (J, φ, μ) take the following form:

$$\Delta = \Delta^i(J_i) \frac{\partial}{\partial \varphi^i}, \quad \omega = \sum_{k=1}^r f_k(J_k) dJ_k \wedge d\varphi^k + \sum_{\alpha, \beta=2r+1}^{2n} f_{\alpha\beta}(\mu_\alpha, \mu_\beta) d\mu_\alpha \wedge d\mu_\beta.$$

In this case, the family of symplectic structures with respect to which Δ is Hamiltonian is exhaustively described in [6,7,15]. The tensor field T can be used to generate compatible invariant symplectic structures according to

$$\omega_T(X, Y) = \omega_1(TX, Y) + \omega_1(X, TY) + \omega_2(X, Y)$$

with

$$\omega_1 = \sum_{k=1}^r f_k(J_k) dJ_k \wedge d\varphi^k, \quad \omega_2 = \frac{1}{2} \sum_{\alpha, \beta=2r+1}^{2n} f_{\alpha\beta}(\mu_\alpha, \mu_\beta) d\mu_\alpha \wedge d\mu_\beta.$$

3.1.1. From noncommutative integrability to invariant tensor fields

Let us suppose that we have a noncommutative integrable system according to Theorem 4. By the integrability analysis, we have the symplectic structure $\omega = d\lambda_i \wedge d\chi^i + dp_\alpha \wedge dq^\alpha$ and the equations of motion

$$\dot{\lambda}_i = 0, \quad \dot{\chi}^i = v_i, \quad \dot{p}_\alpha = 0, \quad \dot{q}^\alpha = 0, \quad 1 \leq i \leq r, \quad r+1 \leq \alpha \leq n,$$

or, calling μ the collection of the p 's and q 's, more simply

$$\dot{\lambda}_i = 0, \quad \dot{\chi}^i = v_i, \quad \dot{\mu}_\alpha = 0.$$

It is easily verified that the following tensor field

$$T = \sum_{j=1}^r \lambda_j \left(\frac{\partial}{\partial \lambda_i} \otimes d\lambda_i + \frac{\partial}{\partial \chi^i} \otimes d\chi^i \right) + C_\rho^\sigma(\mu) \frac{\partial}{\partial \mu_\rho} \otimes d\mu_\sigma.$$

is invariant and, for all diagonalizable matrix $C_\rho^\sigma(\mu) = \delta_\rho^\sigma \mu_\sigma$, has a vanishing torsion, provided that the Hamiltonian function is separable in the form

$$H = K_1(\lambda) + K_2(\mu)$$

with

$$K_1(\lambda) = \sum_{i=1}^r H_i(\lambda_i).$$

If K_1 is not separable but

$$\det \left(\frac{\partial^2 K_1}{\partial \lambda_j \partial \lambda_i} \right) \neq 0,$$

the construction of the invariant tensor field follows strictly the lines of Section 2.1.

This shows that also in the noncommutative case, an invariant torsionless tensor field can always be found. Of course, such a tensor field always generates, by repeated application, Abelian algebras of symmetries. Regardless of the vanishing of torsion on the whole space, the noncommutative features are linked to the nondegenerate eigenvalues and, then, are still described by the term $C_\rho^\sigma(\mu)(\partial/\partial\mu_\rho) \otimes d\mu_\sigma$.

4. Example: the Kepler dynamics

4.1. A recursion operator in the commutative case

The vector field for the Kepler problem, in spherical–polar coordinates, for $\mathfrak{R}^3 - \{0\}$, is globally Hamiltonian with respect to the symplectic form

$$\omega = \sum_i dp_i \wedge dq^i, \quad i = r, \vartheta, \varphi, \tag{18}$$

with Hamiltonian H given by

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \vartheta} \right) + V(r), \quad V(r) = -\frac{k}{r}. \tag{19}$$

In action–angle coordinates (J, φ) , the Kepler Hamiltonian H , the symplectic form ω and the vector field Δ become

$$H = -\frac{mk^2}{(J_r + J_\vartheta + J_\varphi)^2}, \quad \omega = \sum_h dJ_h \wedge d\varphi^h, \tag{20}$$

$$\Delta = \frac{2mk^2}{(J_r + J_\vartheta + J_\varphi)^3} \left(\frac{\partial}{\partial\varphi^1} + \frac{\partial}{\partial\varphi^2} + \frac{\partial}{\partial\varphi^3} \right).$$

It has been shown [29] that the vector field Δ is globally Hamiltonian also with respect to the symplectic form ω_1 :

$$\omega_1 = \sum_{hk} S_k^h dJ_h \wedge d\varphi^k, \tag{21}$$

where the matrix S is defined by

$$S = \frac{1}{2} \begin{vmatrix} J_1 & J_2 & J_3 \\ J_2 - J_3 & J_1 + J_3 & J_3 \\ J_3 - J_2 & J_2 & J_1 + J_2 \end{vmatrix}.$$

We have

$$\Delta = \{H_1, \}_1, \tag{22}$$

with Hamiltonian H_1 given by

$$H_1 = -\frac{2mk^2}{J_r + J_\vartheta + J_\varphi} \tag{23}$$

and the new Poisson brackets

$$\{f, g\}_1 = \sum_{hk} (S^{-1})_h^k \left(\frac{\partial f}{\partial J_h} \frac{\partial g}{\partial \varphi^k} - \frac{\partial f}{\partial \varphi^k} \frac{\partial g}{\partial J_h} \right). \quad (24)$$

In the original coordinates (p, q) , the symplectic form ω is simply written as

$$\omega_1 = \sum_i dK_i \wedge d\alpha^i, \quad (25)$$

where the functions $K_i(p, q)$ and $\alpha^i(p, q)$, defined by

$$\begin{aligned} K_1 &= \frac{1}{4}[J_1^2 + (J_2 - J_3)^2](p, q), & K_2 &= \frac{1}{2}J_2(J_1 + J_3)(p, q), \\ K_3 &= \frac{1}{2}J_3[J_1 + J_2](p, q), & \alpha^i &= \varphi^i(p, q), \end{aligned}$$

are considered as functions of p, q by means of the map $J_i = J_i(p, q)$, $\varphi^i = \varphi^i(p, q)$. As a consequence, a mixed invariant tensor field T defined for nondegenerate ω by $\omega(\hat{T}X, Y) = \omega_1(X, Y)$ can be constructed.

The tensor field

$$T = \sum_{hk} \left(S_k^h dJ_h \otimes \frac{\partial}{\partial J_k} + (S^+)_h^k d\varphi^h \otimes \frac{\partial}{\partial \varphi^k} \right) \quad (26)$$

has double degenerate eigenvalues and vanishing Nijenhuis torsion, the last property being equivalent to the compatibility of the symplectic structures ω and ω_1 .

4.2. A recursion operator in the noncommutative case

The Kepler dynamics has five first integrals given by the components of the angular momentum and the components of the orthogonal Laplace–Runge–Lenz vector.

In action–angle coordinates (J/φ) such first integrals are given by

$$J_1, J_2, J_3, \varphi_1 - \varphi_2, \varphi_2 - \varphi_3.$$

By using the Delauney action–angle coordinates

$$\begin{aligned} I_1 &= J_1 + J_2 + J_3 \equiv \lambda_1, & I_2 &= J_2 + J_3 \equiv \mu_3, & I_3 &= J_3 \equiv \mu_4, \\ \alpha_1 &= \varphi_1 \equiv \chi_1, & \alpha_2 &= \varphi_2 - \varphi_1 \equiv \mu_5, & \alpha_3 &= \varphi_3 - \varphi_2 \equiv \mu_6, \end{aligned}$$

we can construct the invariant torsionless tensor field

$$T = \lambda_1 \left(\frac{\partial}{\partial \lambda_1} \otimes d\lambda_1 + \frac{\partial}{\partial \chi_1} \otimes d\chi_1 \right) + \sum_{\alpha=3}^6 \mu_\alpha \frac{\partial}{\partial \mu_\alpha} \otimes d\mu_\alpha.$$

5. Conclusion

It has been shown that also in the noncommutative case a criterion of integrability can be formulated in terms of invariant “semitorsionless” (1,1) tensor field in close analogy with

the commutative case. Moreover, it has also been shown that in such cases a new invariant $(1,1)$ tensor field can be constructed with a vanishing Nijenhuis torsion. By using either of them, sequences of compatible symplectic structures can be constructed.

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